

PITHAPUR RAJAH'S GOVERNMENT COLLEGE ,KAKINADA

DEPARTMENT OF MATHEMATICS



Laveti. Surya Bala Ratna Bhanu M.Sc ,B.Ed

Lecturer in Mathematics

Phone:7330946793 ,9704768781.

Email: bhanuasdgdc@gmail.com

Paper III - Abstract Algebra**Unit I - Groups****1.8 – Order of an element of a group (Index: 2.3.3.1.8)****Handout**

Order of an element of a group: Let $(G, \cdot)$ be a group and a be any element of G .

Then the order of the element a is defined as the least positive integer n such that $a^n = e$.

If there exists no positive integer n such that $a^n = e$, then we say that a is of infinite order or zero order.

We denote the order of a by $O(a)$ or $|a|$.

Example:

- If $G = \{1, -1\}$ then G is a finite group under usual multiplication. Here $O(1) = 1$ & $O(-1) = 2$ since $(-1)^2 = 1$.
- If $G = \{1, \omega, \omega^2\}$ then G is a finite group under usual multiplication. Here $O(1) = 1$ & $O(\omega) = 3$ ($\because (\omega)^3 = 1$ and $O(\omega^2) = 3$ ($\because (\omega^2)^3 = 1$))

POINTS TO REMEMBER:

- The order of every element of a finite group is finite and is less than or equal to the order of the group.
- In a group G , if $a \in G$, then $O(a) = O(a^{-1})$.
- If a is an element of a group G such that $O(a) = n$ then

$$a^m = e \text{ iff } n/m$$

Problem : The order of every element of a finite group is finite and is less than or equal to the order of the group.

Proof : Let $(G, \cdot)$ be a finite group, let $a \in G$, by closure law we have $a^2, a^3, a^4, \dots \in G$ since G is finite, all the positive integral powers of a .

Let $a^r = a^s$ where $r, s \in \mathbb{N}$ and $r > s$

$$\therefore a^r = a^s \Rightarrow a^{r-s} = a^0 = e \Rightarrow a^m = e \text{ where } r-s=m$$

Since $r > s$, m is a positive integers

$$\therefore \exists \text{ a positive integers } m \text{ such that } a^m = e$$

Hence if n is the positive integral value of m such that $a^n = e$, then $O(a) = n$

$$\therefore O(a) \text{ is finite.}$$

Now to prove that $O(a) \leq O(G)$

If possible let $O(a) > O(G)$.

Let $O(a) = n$ by closure we have $a^2, a^3, a^4, \dots \in G$

No those two elements are equal, for if possible Let $a^r = a^s$ then $a^{r-s} = e$ since

$$0 < r - s < n, O(a) < n \text{ which is contradiction}$$

Hence the n elements $a^2, a^3, a^4, \dots \in G$ are distinct elements of G

Hence $O(a) \leq O(G)$

Theorem: The order of any positive integral power of an element a in a group G cannot exceed the order of a i.e in a group $G, O(a^m) \leq O(a), a \in G$ and $m \in \mathbb{N}$.

Proof : Let $O(a) = n$. if m is a positive integer, then $a^m \in G, O(a) = n, a^n =$

$$e \Rightarrow (a^n)^m = e^m \Rightarrow a^{mn} = e \Rightarrow (a^m)^n = e$$

$$\therefore O(a^m) \leq n \Rightarrow O(a^m) \leq O(a)$$

Exercise Problems:

- For any two elements $a, b \in G$ where G is a group then $O(a) = O(b^{-1}ab)$
- Show that all groups of order 4 and less are commutative.
- If a is an element of a group G such that $O(a) = n$, then $a^m = e$ iff n/m

References:

1. A Text Book of B.Sc. Mathematics, Vol-II, S. Chand & Co.
2. Abstract Algebra by J.B. Fraleigh
3. <https://prgc.ac.in/departments-5-studyresources>
4. <http://ccelms.ap.gov.in/rusa/>